Proposition 6.4 (Class 14)

June 19, 2015

```
# parametrize the Hadamard square root of rank five


#[
# [  1,  1,   1,   1,   0,   0,   0,   0],
# [  1, y3,   0,   0,  y7,   0,   0,   0],
# [  0, y4,   1,  y5,   0, y10,   0,   0],
# [  0,  1,   0,   0,  y8, y11,   0,   0],
# [ y1,  0,   0,   1,   0,   0,   1,   0],
# [  0,  0,   0,  y6,   0,   1,   1,   0],
# [ y2,  0,   1,   0,   0,   0,   0,   1],
# [  0,  0,   1,   0,   0, y12,   0, y16],
# [  1,  0,   0,   0,   1,   0, y14, y17],
# [  0,  0,   0,   0,  y9, y14, y15,   1]]



R.<y1,y2,y3,y4,y5,y6,y7,y8,y9,y10,y11,y12,y13,y14,y15,y16,y17>=QQ[];

M = matrix(R,[
 [  1,  1,   1,   1,   0,   0,   0,   0],
 [  1, y3,   0,   0,  y7,   0,   0,   0],
 [  0, y4,   1,  y5,   0, y10,   0,   0],
 [  0,  1,   0,   0,  y8, y11,   0,   0],
 [ y1,  0,   0,   1,   0,   0,   1,   0],
 [  0,  0,   0,  y6,   0,   1,   1,   0],
 [ y2,  0,   1,   0,   0,   0,   0,   1],
 [  0,  0,   1,   0,   0, y12,   0, y16],
 [  1,  0,   0,   0,   1,   0, y14, y17],
 [  0,  0,   0,   0,  y9, y13, y15,   1]]); M

J=ideal(M.minors(6));
JJ=ideal(y1*y2*y3*y4*y5*y6*y7*y8*y9*y10*y11*y12*y13*y14*y15*y16*y17)
KK=J.saturation(JJ)
KK
[  1   1   1   1   0   0   0   0]
[  1  y3   0   0  y7   0   0   0]
[  0  y4   1  y5   0 y10   0   0]
[  0   1   0   0  y8 y11   0   0]
```

```
[ y1   0   0   1   0   0   1   0]
[  0   0   0  y6   0   1   1   0]
[ y2   0   1   0   0   0   0   1]
[  0   0   1   0   0 y12   0 y16]
[  1   0   0   0   1   0 y14 y17]
[  0   0   0   0  y9 y13 y15   1]
(Ideal (y16 - 1, y15 - 1, y14 - y17, y10 - y11 - y12 + y13 - 1, y8 - y9, y6 - 1, y5 - 1,
y4 - 1, y13*y17 - y11 - y12 + y13 - 1, y9*y17 - 1, y7*y17 - y3, y1*y13 - y9, y9*y12 -
y2*y13, y7*y12 - y3*y13 - y7*y13 + y7 + y13, y1*y12 - y2, y9*y11 + y2*y13 - y9*y13 + y9 -
y13, y7*y11 - y13, y3*y11 - y11 - y12 + y13 - 1, y2*y11 + y2*y12 - y2*y13 + y2 - y12,
y1*y11 + y1 + y2 - y9 - 1, y3*y9 - y7, y1*y7 + y2*y7 - y7*y9 - y7 + y9, y1*y3 + y2*y3 - y3
- y7 + 1, y2*y7*y13 - y7*y9*y13 + y7*y9 - y7*y13 + y9*y13, y2*y3*y13 - y3*y13 - y7*y13 +
y7 + y13, y2*y3*y12 + y2*y3 - y3*y12 - y3*y13 - y7*y13 + y7 + y12 + y13) of Multivariate
Polynomial Ring in y1, y2, y3, y4, y5, y6, y7, y8, y9, y10, y11, y12, y13, y14, y15, y16,
y17 over Rational Field, 0)
```

```
# y16 - 1, y15 - 1, y14 - y17, y8 - y9, y6 - 1, y4 - 1, y5 - 1

R.<y1,y2,y3,y7,y8,y10,y11,y12,y13,y14>=QQ[];

M = matrix(R,[
 [  1,  1,   1,   1,   0,   0,   0,   0],
 [  1, y3,   0,   0,  y7,   0,   0,   0],
 [  0,  1,   1,   1,   0, y10,   0,   0],
 [  0,  1,   0,   0,  y8, y11,   0,   0],
 [ y1,  0,   0,   1,   0,   0,   1,   0],
 [  0,  0,   0,   1,   0,   1,   1,   0],
 [ y2,  0,   1,   0,   0,   0,   0,   1],
 [  0,  0,   1,   0,   0, y12,   0,   1],
 [  1,  0,   0,   0,   1,   0, y14, y14],
 [  0,  0,   0,   0,  y8, y13,   1,   1]]); M

J=ideal(M.minors(6));
JJ=ideal(y1*y2*y3*y7*y8*y10*y11*y12*y13*y14)
KK=J.saturation(JJ)
KK
```

```
[  1   1   1   1   0   0   0   0]
[  1  y3   0   0  y7   0   0   0]
[  0   1   1   1   0 y10   0   0]
[  0   1   0   0  y8 y11   0   0]
[ y1   0   0   1   0   0   1   0]
[  0   0   0   1   0   1   1   0]
[ y2   0   1   0   0   0   0   1]
[  0   0   1   0   0 y12   0   1]
[  1   0   0   0   1   0 y14 y14]
[  0   0   0   0  y8 y13   1   1]
(Ideal (y10 - y11 - y12 + y13 - 1, y13*y14 - y11 - y12 + y13 - 1, y8*y14 - 1, y7*y14 - y3,
y1*y13 - y8, y8*y12 - y2*y13, y7*y12 - y3*y13 - y7*y13 + y7 + y13, y1*y12 - y2, y8*y11 +
y2*y13 - y8*y13 + y8 - y13, y7*y11 - y13, y3*y11 - y11 - y12 + y13 - 1, y2*y11 + y2*y12 -
y2*y13 + y2 - y12, y1*y11 + y1 + y2 - y8 - 1, y3*y8 - y7, y1*y7 + y2*y7 - y7*y8 - y7 + y8,
```

```
y1*y3 + y2*y3 - y3 - y7 + 1, y2*y7*y13 - y7*y8*y13 + y7*y8 - y7*y13 + y8*y13, y2*y3*y13 -
y3*y13 - y7*y13 + y7 + y13, y2*y3*y12 + y2*y3 - y3*y12 - y3*y13 - y7*y13 + y7 + y12 + y13)
of Multivariate Polynomial Ring in y1, y2, y3, y7, y8, y10, y11, y12, y13, y14 over
Rational Field, 0)
```

```
#  y8*y14 - 1
#  y7*y14 - y3 => y7 = y3*y8

R.<y1,y2,y3,y8,y10,y11,y12,y13>=QQ[];

M = matrix(R,[
 [  1,  1,   1,   1,   0,   0,   0,   0],
 [  1, y3,   0,   0,  y3,   0,   0,   0],
 [  0,  1,   1,   1,   0, y10,   0,   0],
 [  0,  1,   0,   0,   1, y11,   0,   0],
 [ y1,  0,   0,   1,   0,   0,   1,   0],
 [  0,  0,   0,   1,   0,   1,   1,   0],
 [ y2,  0,   1,   0,   0,   0,   0,   1],
 [  0,  0,   1,   0,   0, y12,   0,   1],
 [ y8,  0,   0,   0,   1,   0,   1,   1],
 [  0,  0,   0,   0,   1, y13,   1,   1]]); M

J=ideal(M.minors(6));
JJ=ideal(y1*y2*y3*y8*y10*y11*y12*y13)
KK=J.saturation(JJ)
KK
```

```
[  1   1   1   1   0   0   0   0]
[  1  y3   0   0  y3   0   0   0]
[  0   1   1   1   0 y10   0   0]
[  0   1   0   0   1 y11   0   0]
[ y1   0   0   1   0   0   1   0]
[  0   0   0   1   0   1   1   0]
[ y2   0   1   0   0   0   0   1]
[  0   0   1   0   0 y12   0   1]
[ y8   0   0   0   1   0   1   1]
[  0   0   0   0   1 y13   1   1]
(Ideal (y10 - y11 - y12 + y13 - 1, y1*y13 - y8, y8*y12 - y2*y13, y1*y12 - y2, y8*y11 +
y2*y13 - y8*y13 + y8 - y13, y3*y11 - y11 - y12 + y13 - 1, y2*y11 + y2*y12 - y2*y13 + y2 -
y12, y1*y11 + y1 + y2 - y8 - 1, y1*y3 + y2*y3 - y3*y8 - y3 + 1, y2*y3*y13 - y3*y8*y13 +
y3*y8 - y3*y13 + y13, y2*y3*y12 - y3*y8*y13 + y2*y3 + y3*y8 - y3*y12 - y3*y13 + y12 + y13)
of Multivariate Polynomial Ring in y1, y2, y3, y8, y10, y11, y12, y13 over Rational Field,
0)
```

```
#  y1*y13 - y8
#  y1*y12 - y2

R.<y1,y3,y10,y11,y12,y13>=QQ[];

M = matrix(R,[
 [     1,  1,   1,   1,   0,   0,   0,   0],
 [     1, y3,   0,   0,  y3,   0,   0,   0],
 [     0,  1,   1,   1,   0, y10,   0,   0],
```

```
 [      0,   1,    0,    0,    1, y11,    0,    0],
 [     y1,   0,    0,    1,    0,   0,    1,    0],
 [      0,   0,    0,    1,    0,   1,    1,    0],
 [y1*y12,   0,    1,    0,    0,   0,    0,    1],
 [      0,   0,    1,    0,    0, y12,    0,    1],
 [y1*y13,   0,    0,    0,    1,   0,    1,    1],
 [      0,   0,    0,    0,    1, y13,    1,    1]]); M

J=ideal(M.minors(6));
JJ=ideal(y1*y3*y10*y11*y12*y13)
KK=J.saturation(JJ)
KK
```

```
[     1      1      1      1      0      0      0      0]
[     1     y3      0      0     y3      0      0      0]
[     0      1      1      1      0    y10      0      0]
[     0      1      0      0      1    y11      0      0]
[    y1      0      0      1      0      0      1      0]
[     0      0      0      1      0      1      1      0]
[y1*y12      0      1      0      0      0      0      1]
[     0      0      1      0      0    y12      0      1]
[y1*y13      0      0      0      1      0      1      1]
[     0      0      0      0      1    y13      1      1]
(Ideal (y10 - y11 - y12 + y13 - 1, y3*y11 - y11 - y12 + y13 - 1, y1*y11 + y1*y12 - y1*y13
+ y1 - 1, y1*y3*y12 - y1*y3*y13 + y1*y3 - y3 + 1) of Multivariate Polynomial Ring in y1,
y3, y10, y11, y12, y13 over Rational Field, 0)
```

```
#  y10 - y11 - y12 + y13 - 1, y3*y11 - y11 - y12 + y13 - 1 => y10 = y3*\
   y11
#  y10 - y11 - y12 + y13 - 1, y1*y11 + y1*y12 - y1*y13 + y1 - 1 => y1*y10\
    = 1

R.<y10,y11,y12,y13>=QQ[];

M = matrix(R,[
 [    y10,   1,    1,    1,    0,      0,    0,    0],
 [    y11,   1,    0,    0,    1,      0,    0,    0],
 [      0,   1,    1,    1,    0,    y10,    0,    0],
 [      0,   1,    0,    0,    1,    y11,    0,    0],
 [      1,   0,    0,    1,    0,      0,    1,    0],
 [      0,   0,    0,    1,    0,      1,    1,    0],
 [    y12,   0,    1,    0,    0,      0,    0,    1],
 [      0,   0,    1,    0,    0,    y12,    0,    1],
 [    y13,   0,    0,    0,    1,      0,    1,    1],
 [      0,   0,    0,    0,    1,    y13,    1,    1]]); M

J=ideal(M.minors(6));
JJ=ideal(y10*y11*y12*y13)
KK=J.saturation(JJ)
KK
```

```
[y10   1   1   1   0   0   0   0]
[y11   1   0   0   1   0   0   0]
```

```
[  0   1   1   1   0 y10   0   0]
[  0   1   0   0   1 y11   0   0]
[  1   0   0   1   0   0   1   0]
[  0   0   0   1   0   1   1   0]
[y12   0   1   0   0   0   0   1]
[  0   0   1   0   0 y12   0   1]
[y13   0   0   0   1   0   1   1]
[  0   0   0   0   1 y13   1   1]
(Ideal (y10 - y11 - y12 + y13 - 1) of Multivariate Polynomial Ring in y10, y11, y12, y13
over Rational Field, 0)
```

```
#  y10 - y11 - y12 + y13 - 1 => y13 = y11 + y12 + 1 - y10

# the Hadamard square root and the slack matrix can  be parametrized in \
   the following way

M = matrix(R,[
 [                  y10,  1,   1,   1,   0,                    0,   0,   \
   0],
 [                  y11,  1,   0,   0,   1,                    0,   0,   \
   0],
 [                    0,  1,   1,   1,   0,                  y10,   0,   \
   0],
 [                    0,  1,   0,   0,   1,                  y11,   0,   \
   0],
 [                    1,  0,   0,   1,   0,                    0,   1,   \
   0],
 [                    0,  0,   0,   1,   0,                    1,   1,   \
   0],
 [                  y12,  0,   1,   0,   0,                    0,   0,   \
   1],
 [                    0,  0,   1,   0,   0,                  y12,   0,   \
   1],
 [ y11 + y12 + 1 - y10,  0,   0,   0,   1,                    0,   1,   \
   1],
 [                    0,  0,   0,   0,   1, y11 + y12 + 1 - y10,   1,   \
   1]]);
```

```
# thus x10 = y10^2, x11 = y11^2, x12 = y12^2, (1 - y10 + y11 + y12)^2 = 1\
   - x10 + x11 + x12)

%macaulay2
R = QQ[y10,y11,y12,x10,x11,x12,MonomialOrder => Eliminate 3]
I = ideal(x10-y10^2,x11-y11^2,x12-y12^2,(1-y10+y11+y12)^2-1+x10-x11-x12)
G = selectInSubring(1,gens gb I);
newpoly = G_0_0;
toString factor newpoly
R

PolynomialRing

          2           2           2          2              2
```

```
2
ideal (- y10  + x10, - y11  + x11, - y12  + x12, y10  - 2y10*y11 + y11  - 2y10*y12 +
2y11*y12 + y12  - 2y10 + 2y11 + 2y12 + x10 - x11 - x12)

Ideal of R

        1       1
Matrix R  <--- R


(x10^8-4*x10^7*x11+6*x10^6*x11^2-4*x10^5*x11^3+x10^4*x11^4-4*x10^7*x12+16*x10^6*x11*x12-24
*x10^5*x11^2*x12+16*x10^4*x11^3*x12-4*x10^3*x11^4*x12+6*x10^6*x12^2-24*x10^5*x11*x12^2+36*
x10^4*x11^2*x12^2-24*x10^3*x11^3*x12^2+6*x10^2*x11^4*x12^2-4*x10^5*x12^3+16*x10^4*x11*x12^
3-24*x10^3*x11^2*x12^3+16*x10^2*x11^3*x12^3-4*x10*x11^4*x12^3+x10^4*x12^4-4*x10^3*x11*x12^
4+6*x10^2*x11^2*x12^4-4*x10*x11^3*x12^4+x11^4*x12^4-4*x10^7+16*x10^6*x11-24*x10^5*x11^2+16
*x10^4*x11^3-4*x10^3*x11^4+16*x10^6*x12-60*x10^5*x11*x12+76*x10^4*x11^2*x12-36*x10^3*x11^3
*x12+4*x10^2*x11^4*x12-24*x10^5*x12^2+76*x10^4*x11*x12^2-76*x10^3*x11^2*x12^2+20*x10^2*x11
^3*x12^2+4*x10*x11^4*x12^2+16*x10^4*x12^3-36*x10^3*x11*x12^3+20*x10^2*x11^2*x12^3+4*x10*x1
1^3*x12^3-4*x11^4*x12^3-4*x10^3*x12^4+4*x10^2*x11*x12^4+4*x10*x11^2*x12^4-4*x11^3*x12^4+6*
x10^6-24*x10^5*x11+36*x10^4*x11^2-24*x10^3*x11^3+6*x10^2*x11^4-24*x10^5*x12+76*x10^4*x11*x
12-76*x10^3*x11^2*x12+20*x10^2*x11^3*x12+4*x10*x11^4*x12+36*x10^4*x12^2-76*x10^3*x11*x12^2
+82*x10^2*x11^2*x12^2-32*x10*x11^3*x12^2+6*x11^4*x12^2-24*x10^3*x12^3+20*x10^2*x11*x12^3-3
2*x10*x11^2*x12^3+4*x11^3*x12^3+6*x10^2*x12^4+4*x10*x11*x12^4+6*x11^2*x12^4-4*x10^5+16*x10
^4*x11-24*x10^3*x11^2+16*x10^2*x11^3-4*x10*x11^4+16*x10^4*x12-36*x10^3*x11*x12+20*x10^2*x1
1^2*x12+4*x10*x11^3*x12-4*x11^4*x12-24*x10^3*x12^2+20*x10^2*x11*x12^2-32*x10*x11^2*x12^2+4
*x11^3*x12^2+16*x10^2*x12^3+4*x10*x11*x12^3+4*x11^2*x12^3-4*x10*x12^4-4*x11*x12^4+x10^4-4*
x10^3*x11+6*x10^2*x11^2-4*x10*x11^3+x11^4-4*x10^3*x12+4*x10^2*x11*x12+4*x10*x11^2*x12-4*x1
1^3*x12+6*x10^2*x12^2+4*x10*x11*x12^2+6*x11^2*x12^2-4*x10*x12^3-4*x11*x12^3+x12^4)
```