Proposition 7.3 (Class 22)

June 19, 2015

```
# parametrize the Hadamard square root of rank five

#[
# [1,  1,  1,  0,  0,  0,  0,  0,  0],
# [1, y1,  0,  1,  0,  1,  0,  1,  0],
# [1,  0,  0, y4,  0, y8,  0,  0,  1],
# [0,  0,  0,  1,  0,  0,y11,y14,  0],
# [1,  0, y2, y5,  0,  0,y12,  0,y17],
# [0,  1,  0,  0,  1, y9,  0,y15,  0],
# [0,  0,  0,  0,  1,y10,  0,  0,y18],
# [0,  1, y3,  0, y6,  0,y13,y16,  0],
# [0,  0,  1,  0, y7,  0,  1,  0,y19]]



R.<y1,y2,y3,y4,y5,y6,y7,y8,y9,y10,y11,y12,y13,y14,y15,y16,y17,y18,y19>=QQ\
    [];

M = matrix(R,[
 [1,  1,  1,  0,  0,  0,  0,  0,  0],
 [1, y1,  0,  1,  0,  1,  0,  1,  0],
 [1,  0,  0, y4,  0, y8,  0,  0,  1],
 [0,  0,  0,  1,  0,  0,y11,y14,  0],
 [1,  0, y2, y5,  0,  0,y12,  0,y17],
 [0,  1,  0,  0,  1, y9,  0,y15,  0],
 [0,  0,  0,  0,  1,y10,  0,  0,y18],
 [0,  1, y3,  0, y6,  0,y13,y16,  0],
 [0,  0,  1,  0, y7,  0,  1,  0,y19]]); M

J=ideal(M.minors(6));
JJ=ideal(y1*y2*y3*y4*y5*y6*y7*y8*y9*y10*y11*y12*y13*y14*y15*y16*y17*y18*\
   y19)
KK=J.saturation(JJ)
KK
[  1   1   1   0   0   0   0   0   0]
[  1  y1   0   1   0   1   0   1   0]
[  1   0   0  y4   0  y8   0   0   1]
[  0   0   0   1   0   0 y11 y14   0]
[  1   0  y2  y5   0   0 y12   0 y17]
```

```
[  0   1   0   0   1  y9   0 y15   0]
[  0   0   0   0   1 y10   0   0 y18]
[  0   1  y3   0  y6   0 y13 y16   0]
[  0   0   1   0  y7   0   1   0 y19]
(Ideal (y16 - y18, y15 - y18, y14 - y18, y11 - 1, y5 - 1, y4 - 1, y3 - y13, y2 - y12,
y12*y19 + y13*y19 - y17 - y19, y1*y19 + y1 - y7 - 1, y12*y18 - y12 - y17 + 1, y9*y18 -
y10*y19 - y9 + y18, y8*y18 + y9 - y10 - y18, y7*y18 + y18 - y19 - 1, y6*y18 + y13*y18 -
y13 - y18, y1*y18 - 1, y9*y17 - y10*y17 + y12 + y17 - y18 - 1, y8*y17 - y12 - y17 + 1,
y7*y17 - y6*y19, y7*y13 - y6*y19 - y13*y19 - y6 + y19 + 1, y1*y13 - y6 - y13 + 1, y8*y12 +
y9*y12 - y10*y12 - y8, y7*y12 + y6*y19 + y13*y19 - y7 - y19 - 1, y6*y12 + y6*y17 + y13*y17
- y6 - y12 - y13 - y17 + 1, y1*y12 + y1*y17 - y1 - y12, y7*y10 - y8 - y9 + y10, y6*y10 -
y8*y13 - y9*y13 + y10*y13 + y9 - y10, y7*y9 + y8*y19 - y19 - 1, y6*y9 - y13, y1*y9 -
y1*y10 + y8 - 1, y13*y18*y19 - y17*y18 - y13*y19 + y17*y19 - y18*y19 + y17, y10*y13*y19 -
y10*y19 - y9 + y18, y8*y13*y19 + y9*y13*y19 - y9*y19 - y9, y6*y13*y19 - y6*y17*y19 -
y13*y17*y19 - y6*y17 + y17*y19 + y17, y8*y10*y19 + y8*y9 + y9^2 - y9*y10 - y10*y19 - y10,
y6*y8*y19 + y13*y19 - y19 - 1, y10*y17*y18 - y10*y17*y19 - y10*y17 + y18^2, y10*y13*y17 -
y9*y12 - y9*y13 - y10*y17 + y13*y18 + y9 + y12 + y17 - y18 - 1, y8*y9*y13 + y9^2*y13 -
y9*y10*y13 - y9^2 + y9*y10 - y10*y13, y9^2*y12 - y9*y10*y12 + y9^2*y13 - y9*y10*y13 +
y10*y12*y13 - y9^2 + y9*y10 - y10*y13, y1*y6*y8 - y6*y7*y8 - y6*y8 - y6*y19 - y13*y19 + y7
+ y19 + 1) of Multivariate Polynomial Ring in y1, y2, y3, y4, y5, y6, y7, y8, y9, y10,
y11, y12, y13, y14, y15, y16, y17, y18, y19 over Rational Field, 0)
```

```
# y16 - y18, y15 - y18, y14 - y18, y11 - 1, y5 - 1, y4 - 1, y3 - y13, y2 \
   - y12


R.<y1,y2,y3,y6,y7,y8,y9,y10,y14,y17,y19>=QQ[];

M = matrix(R,[
 [1,  1,  1,  0,  0,  0,  0,  0,  0],
 [1, y1,  0,  1,  0,  1,  0,  1,  0],
 [1,  0,  0,  1,  0, y8,  0,  0,  1],
 [0,  0,  0,  1,  0,  0,  1,y14,  0],
 [1,  0, y2,  1,  0,  0, y2,  0,y17],
 [0,  1,  0,  0,  1, y9,  0,y14,  0],
 [0,  0,  0,  0,  1,y10,  0,  0,y14],
 [0,  1, y3,  0, y6,  0, y3,y14,  0],
 [0,  0,  1,  0, y7,  0,  1,  0,y19]]); M

J=ideal(M.minors(6));
JJ=ideal(y1*y2*y3*y6*y7*y8*y9*y10*y14*y17*y19)
KK=J.saturation(JJ)
KK
```

```
[  1   1   1   0   0   0   0   0   0]
[  1  y1   0   1   0   1   0   1   0]
[  1   0   0   1   0  y8   0   0   1]
[  0   0   0   1   0   0   1 y14   0]
[  1   0  y2   1   0   0  y2   0 y17]
[  0   1   0   0   1  y9   0 y14   0]
[  0   0   0   0   1 y10   0   0 y14]
```

```
[  0   1  y3   0  y6   0  y3 y14   0]
[  0   0   1   0  y7   0   1   0 y19]
(Ideal (y2*y19 + y3*y19 - y17 - y19, y1*y19 + y1 - y7 - 1, y9*y17 - y10*y17 + y2 - y14 +
y17 - 1, y8*y17 - y2 - y17 + 1, y7*y17 - y6*y19, y9*y14 - y10*y19 - y9 + y14, y8*y14 + y9
- y10 - y14, y7*y14 + y14 - y19 - 1, y3*y14 + y6*y14 - y3 - y14, y2*y14 - y2 - y17 + 1,
y1*y14 - 1, y7*y10 - y8 - y9 + y10, y7*y9 + y8*y19 - y19 - 1, y6*y9 - y3, y1*y9 - y1*y10 +
y8 - 1, y3*y8 + y3*y9 - y3*y10 - y6*y10 - y9 + y10, y2*y8 + y2*y9 - y2*y10 - y8, y3*y7 -
y3*y19 - y6*y19 - y6 + y19 + 1, y2*y7 + y3*y19 + y6*y19 - y7 - y19 - 1, y2*y6 + y3*y17 +
y6*y17 - y2 - y3 - y6 - y17 + 1, y1*y3 - y3 - y6 + 1, y1*y2 + y1*y17 - y1 - y2, y6*y14*y19
+ y14*y17 - y17*y19 - y17, y8*y10*y19 + y8*y9 + y9^2 - y9*y10 - y10*y19 - y10, y6*y10*y19
- y14, y3*y10*y19 - y10*y19 - y9 + y14, y6*y8*y19 + y3*y19 - y19 - 1, y3*y6*y19 -
y3*y17*y19 - y6*y17*y19 - y6*y17 + y17*y19 + y17, y10*y14*y17 - y10*y17*y19 + y14^2 -
y10*y17, y6*y10*y17 + y6*y14 - y2 - y3 - y17 + 1, y3*y10*y17 - y2*y9 - y3*y9 - y6*y14 -
y10*y17 + y2 + y3 + y9 + y17 - 1, y1*y6*y10 - y6*y8 - y3 + 1, y2*y9^2 + y3*y9^2 +
y2*y3*y10 - y2*y9*y10 - y3*y9*y10 - y9^2 - y3*y10 + y9*y10, y1*y6*y8 - y6*y7*y8 - y6*y8 -
y3*y19 - y6*y19 + y7 + y19 + 1) of Multivariate Polynomial Ring in y1, y2, y3, y6, y7, y8,
y9, y10, y14, y17, y19 over Rational Field, 0)
```

```
# y1*y2 + y1*y17 - y1 - y2 => y1*y17 = 1
# y8*y17 - y2 - y17 + 1 => y8 = y2
# y1*y14 - 1 => y17 = y14
# y6*y14*y19 + y14*y17 - y17*y19 - y17 => y6*y14 = 1 => y6 = y1

R.<y1,y2,y3,y7,y9,y10,y14,y19>=QQ[];

M = matrix(R,[
 [1,  1,  1,  0,  0,  0,  0,  0,  0],
 [1, y1,  0,  1,  0,  1,  0,  1,  0],
 [1,  0,  0,  1,  0, y2,  0,  0,  1],
 [0,  0,  0,  1,  0,  0,  1,y14,  0],
 [1,  0, y2,  1,  0,  0, y2,  0,y14],
 [0,  1,  0,  0,  1, y9,  0,y14,  0],
 [0,  0,  0,  0,  1,y10,  0,  0,y14],
 [0,  1, y3,  0, y1,  0, y3,y14,  0],
 [0,  0,  1,  0, y7,  0,  1,  0,y19]]); M

J=ideal(M.minors(6));
JJ=ideal(y1*y2*y3*y7*y9*y10*y14*y19)
KK=J.saturation(JJ)
KK
```

```
[  1   1   1   0   0   0   0   0   0]
[  1  y1   0   1   0   1   0   1   0]
[  1   0   0   1   0  y2   0   0   1]
[  0   0   0   1   0   0   1 y14   0]
[  1   0  y2   1   0   0  y2   0 y14]
[  0   1   0   0   1  y9   0 y14   0]
[  0   0   0   0   1 y10   0   0 y14]
[  0   1  y3   0  y1   0  y3 y14   0]
[  0   0   1   0  y7   0   1   0 y19]
(Ideal (y3 - y9 + y14 - 1, y2 + y9 - y10 - 1, y1 - y7 - y14 + y19, y10*y19 - y14*y19 - y14
```

```
+ y19, y7*y19 + y14*y19 - y19^2 + y14 - y19 - 1, y14^2 - y14*y19 - y14 + y19, y10*y14 -
y14*y19 - y10 + y19, y9*y14 - y14*y19 - y9 + y19, y7*y14 + y14 - y19 - 1, y7*y10 - 1,
y7*y9 - y9*y19 + y14*y19 + y14 - y19 - 1) of Multivariate Polynomial Ring in y1, y2, y3,
y7, y9, y10, y14, y19 over Rational Field, 0)
```

```
# y3 - y9 + y14 - 1 => y9 = y3*y14
# y2 + y9 - y10 - 1 => y10 = y2*y9 = y2*y3*y14
# y7*y14 + y14 - y19 - 1 => y19 = y14^2*y7

R.<y1,y2,y3,y7,y14>=QQ[];

M = matrix(R,[
 [1,  1,  1,  0,  0,         0,  0,  0,        0],
 [1, y1,  0,  1,  0,         1,  0,  1,        0],
 [1,  0,  0,  1,  0,        y2,  0,  0,        1],
 [0,  0,  0,  1,  0,         0,  1,y14,        0],
 [1,  0, y2,  1,  0,         0, y2,  0,      y14],
 [0,  1,  0,  0,  1,    y3*y14,  0,y14,        0],
 [0,  0,  0,  0,  1,y2*y3*y14,  0,  0,      y14],
 [0,  1, y3,  0, y1,         0, y3,y14,        0],
 [0,  0,  1,  0, y7,         0,  1,  0,y14^2*y7]]); M

J=ideal(M.minors(6));
JJ=ideal(y1*y2*y3*y7*y14)
KK=J.saturation(JJ)
KK
```

```
[       1        1        1        0        0        0        0        0
0]
[       1       y1        0        1        0        1        0        1
0]
[       1        0        0        1        0       y2        0        0
1]
[       0        0        0        1        0        0        1      y14
0]
[       1        0       y2        1        0        0       y2        0
y14]
[       0        1        0        0        1    y3*y14        0      y14
0]
[       0        0        0        0        1 y2*y3*y14        0        0
y14]
[       0        1       y3        0       y1        0       y3      y14
0]
[       0        0        1        0       y7        0        1        0
y7*y14^2]
(Ideal (y7*y14 + y1 - y7 - 1, y3*y14 - y3 - y14 + 1, y2*y14 - y2 - y14 + 1, y1*y14 - 1,
y2*y7 + y3*y7 - y1 - y7, y2*y3 - y2 - y3 + 1, y1*y3 - y1 - y3 + 1, y1*y2 - y1 - y2 + 1,
y1^2 - y1*y7 - y1 + y7, y3^2*y7 - y3*y7 - y3 + 1) of Multivariate Polynomial Ring in y1,
y2, y3, y7, y14 over Rational Field, 0)
```

```
KK[0].primary_decomposition();
[Ideal (y14 - 1, y3 - 1, y1 - 1, y2*y7 - 1) of Multivariate Polynomial Ring in y1, y2, y3,
```

y7, y14 over Rational Field, Ideal (y14 - 1, y2 - 1, y1 - 1, y3*y7 - 1) of Multivariate Polynomial Ring in y1, y2, y3, y7, y14 over Rational Field, Ideal (y3 - 1, y2 - 1, y1 - y7, y7*y14 - 1) of Multivariate Polynomial Ring in y1, y2, y3, y7, y14 over Rational Field]

```
# results

# for y14 - 1, y3 - 1, y1 - 1, y2*y7 - 1 the slack matrix has the form

M = matrix(R,[
 [1,  1,  1,  0,  0,   0,  0,  0,  0],
 [1,  1,  0,  1,  0,   1,  0,  1,  0],
 [1,  0,  0,  1,  0,  y2,  0,  0,  1],
 [0,  0,  0,  1,  0,   0,  1,  1,  0],
 [1,  0, y2,  1,  0,   0, y2,  0,  1],
 [0,  1,  0,  0,  1,   1,  0,  1,  0],
 [0,  0,  0,  0,  1,  y2,  0,  0,  1],
 [0,  1,  1,  0,  1,   0,  1,  1,  0],
 [0,  0, y2,  0,  1,   0, y2,  0,  1]]);

M.matrix_from_rows_and_columns([0,7,5,3,1,8,6,4,2],[1,2,0,7,8,4,3,6,5])
[ 1  1  1  0  0  0  0  0  0]
[ 1  1  0  1  0  1  0  1  0]
[ 1  0  0  1  0  1  0  0  1]
[ 0  0  0  1  0  0  1  1  0]
[ 1  0  1  1  0  0  1  0  1]
[ 0 y2  0  0  1  1  0 y2  0]
[ 0  0  0  0  1  1  0  0 y2]
[ 0 y2  1  0  1  0  1 y2  0]
[ 0  0  1  0  1  0  1  0 y2]
```

```
# for y14 - 1, y2 - 1, y1 - 1, y3*y7 - 1 the slack matrix has the form

M = matrix(R,[
 [1,  1,  1,  0,  0,  0,  0,  0,  0],
 [1,  1,  0,  1,  0,  1,  0,  1,  0],
 [1,  0,  0,  1,  0,  1,  0,  0,  1],
 [0,  0,  0,  1,  0,  0,  1,  1,  0],
 [1,  0,  1,  1,  0,  0,  1,  0,  1],
 [0,  1,  0,  0,  1, y3,  0,  1,  0],
 [0,  0,  0,  0,  1, y3,  0,  0,  1],
 [0,  1, y3,  0,  1,  0, y3,  1,  0],
 [0,  0, y3,  0,  1,  0, y3,  0,  1]]);


M.matrix_from_rows_and_columns([0,4,2,3,1,8,6,7,5],[0,2,1,3,4,8,7,6,5])
[ 1  1  1  0  0  0  0  0  0]
[ 1  1  0  1  0  1  0  1  0]
[ 1  0  0  1  0  1  0  0  1]
```

```
[ 0  0  0  1  0  0  1  1  0]
[ 1  0  1  1  0  0  1  0  1]
[ 0 y3  0  0  1  1  0 y3  0]
[ 0  0  0  0  1  1  0  0 y3]
[ 0 y3  1  0  1  0  1 y3  0]
[ 0  0  1  0  1  0  1  0 y3]
```

```
# for y3 - 1, y2 - 1, y1 - y7, y7*y14 - 1 the slack matrix has the form

M = matrix(R,[
 [1,  1,  1,  0,  0,   0,  0,  0,   0],
 [1, y7,  0,  1,  0,  y7,  0, y7,   0],
 [1,  0,  0,  1,  0,  y7,  0,  0,  y7],
 [0,  0,  0,  1,  0,   0,  1,  1,   0],
 [1,  0,  1,  1,  0,   0,  1,  0,   1],
 [0,  1,  0,  0,  1,   1,  0,  1,   0],
 [0,  0,  0,  0,  1,   1,  0,  0,   1],
 [0,  1,  1,  0, y7,   0,  1,  1,   0],
 [0,  0,  1,  0, y7,   0,  1,  0,   1]]);


(diagonal_matrix([1,1,1,1,1,y7,y7,1,1])*M*diagonal_matrix([1,1,1,1,1/y7\
   ,1/y7,1,1,1])).matrix_from_rows_and_columns\
   ([0,7,8,3,4,5,6,1,2],[2,1,0,6,5,4,3,7,8])
```

```
[ 1  1  1  0  0  0  0  0  0]
[ 1  1  0  1  0  1  0  1  0]
[ 1  0  0  1  0  1  0  0  1]
[ 0  0  0  1  0  0  1  1  0]
[ 1  0  1  1  0  0  1  0  1]
[ 0 y7  0  0  1  1  0 y7  0]
[ 0  0  0  0  1  1  0  0 y7]
[ 0 y7  1  0  1  0  1 y7  0]
[ 0  0  1  0  1  0  1  0 y7]
```