Proposition 7.3 (Class 26)

June 19, 2015

```
# parametrize the Hadamard square root of rank five

# 1)

#[
# [ 1, 0, 1, 0, 1, 0, y7,  0, y7,  0,     y7,      0],
# [ 1, 0, 1, 0, 0, 1, y8,  0, y8,  0,      0,     y8],
# [ 1, 0, 0, 1, 1, 0, y9,  0,  0, y9,     y9,      0],
# [ 1, 0, 0, 1, 0, 1,y10,  0,  0,y10,      0,    y10],
# [ 0, 1, 1, 0,y4, 0,  0,y11,y11,  0,y11*y12,      0],
# [ 0, 1, 1, 0, 0,y4,  0,y13,y13,  0,      0,y13*y12],
# [ 0, 1, 0, 1,y4, 0,  0,y14,  0,y14,y14*y12,      0],
# [ 0, 1, 0, 1, 0,y4,  0,y15,  0,y15,      0,y15*y12],
# [ 0, 0, 0, 0, 0, 0,  1,y16,y17,y18,    y19,    y20],
# [ 1,y1,y2,y3,y5,y6,  0,  0,  0,  0,      0,      0]]


# 2)

#[
# [ 1, 0, 1, 0, 1, 0, y7,  0, y7,  0,     y7,      0],
# [ 1, 0, 1, 0, 0, 1, y8,  0, y8,  0,      0,     y8],
# [ 1, 0, 0, 1, 1, 0, y9,  0,  0, y9, y9*y12,      0],
# [ 1, 0, 0, 1, 0, 1,y10,  0,  0,y10,      0,y10*y12],
# [ 0, 1, 1, 0,y4, 0,  0,y11,y11,  0,    y11,      0],
# [ 0, 1, 1, 0, 0,y4,  0,y13,y13,  0,      0,    y13],
# [ 0, 1, 0, 1,y4, 0,  0,y14,  0,y14,y14*y12,      0],
# [ 0, 1, 0, 1, 0,y4,  0,y15,  0,y15,      0,y15*y12],
# [ 0, 0, 0, 0, 0, 0,  1,y16,y17,y18,    y19,    y20],
# [ 1,y1,y2,y3,y5,y6,  0,  0,  0,  0,      0,      0]]



#  1)

R.<y1,y2,y3,y4,y5,y6,y7,y8,y9,y10,y11,y12,y13,y14,y15,y16,y17,y18,y19,y20\
   >=QQ[];

M = matrix(R,[
 [ 1, 0, 1, 0, 1, 0, y7,  0, y7,  0,     y7,      0],
```

```
 [ 1, 0, 1, 0, 0, 1, y8,  0, y8,  0,      0,     y8],
 [ 1, 0, 0, 1, 1, 0, y9,  0,  0, y9,     y9,      0],
 [ 1, 0, 0, 1, 0, 1,y10,  0,  0,y10,      0,    y10],
 [ 0, 1, 1, 0,y4, 0,  0,y11,y11,  0,y11*y12,      0],
 [ 0, 1, 1, 0, 0,y4,  0,y13,y13,  0,      0,y13*y12],
 [ 0, 1, 0, 1,y4, 0,  0,y14,  0,y14,y14*y12,      0],
 [ 0, 1, 0, 1, 0,y4,  0,y15,  0,y15,      0,y15*y12],
 [ 0, 0, 0, 0, 0, 0,  1,y16,y17,y18,    y19,    y20],
 [ 1,y1,y2,y3,y5,y6,  0,  0,  0,  0,      0,      0]])
K=M.matrix_from_rows([0,1,2,3,4,5,6,7,8])
J=ideal(K.minors(6));
JJ=ideal(y1*y2*y3*y4*y5*y6*y7*y8*y9*y10*y11*y12*y13*y14*y15*y16*y17*y18*\
   y19*y20)
KK=J.saturation(JJ)
KK
```

```
(Ideal (y14 - y15, y13 - y15, y11 - y15, y10 - y15, y9 - y15, y8 - y15, y7 - y15, y4 -
y12) of Multivariate Polynomial Ring in y1, y2, y3, y4, y5, y6, y7, y8, y9, y10, y11, y12,
y13, y14, y15, y16, y17, y18, y19, y20 over Rational Field, 1)
```

```
# 1)

# y14 - y15, y13 - y15, y11 - y15, y10 - y15, y9 - y15, y8 - y15, y7 - \
   y15, y4 - y12

R.<y1,y2,y3,y4,y5,y6,y7,y16,y17,y18,y19,y20>=QQ[];

M = matrix(R,[
 [ 1, 0, 1, 0, 1, 0,y7,  0, y7,  0,   y7,    0],
 [ 1, 0, 1, 0, 0, 1,y7,  0, y7,  0,    0,   y7],
 [ 1, 0, 0, 1, 1, 0,y7,  0,  0, y7,   y7,    0],
 [ 1, 0, 0, 1, 0, 1,y7,  0,  0, y7,    0,   y7],
 [ 0, 1, 1, 0,y4, 0, 0, y7, y7,  0,y7*y4,    0],
 [ 0, 1, 1, 0, 0,y4, 0, y7, y7,  0,    0,y7*y4],
 [ 0, 1, 0, 1,y4, 0, 0, y7,  0, y7,y7*y4,    0],
 [ 0, 1, 0, 1, 0,y4, 0, y7,  0, y7,    0,y7*y4],
 [ 0, 0, 0, 0, 0, 0, 1,y16,y17,y18,  y19,  y20],
 [ 1,y1,y2,y3,y5,y6, 0,  0,  0,  0,    0,    0]])
J=ideal(M.minors(6));
JJ=ideal(y1*y2*y3*y4*y5*y6*y7*y16*y17*y18*y19*y20)
KK=J.saturation(JJ)
KK
```

```
(Ideal (y16 - y17 - y18 + 1, y6 - y20, y5 - y19, y3 - y18, y2 - y17, y1 - y17 - y18 + 1,
y4*y17 + y4*y18 - y4 - y19 - y20 + 1) of Multivariate Polynomial Ring in y1, y2, y3, y4,
y5, y6, y7, y16, y17, y18, y19, y20 over Rational Field, 1)
```

```
# 1)


# y6 - y20, y5 - y19, y3 - y18, y2 - y17
# y16 - y17 - y18 + 1, y1 - y17 - y18 + 1 => y16 = y1
# y1 - y17 - y18 + 1 => y17*y18 = y1 => y1= y2*y3
```

```
R.<y2,y3,y4,y5,y6,y7>=QQ[];

M = matrix(R,[
 [ 1,    0, 1, 0, 1, 0,y7,    0, y7,  0,   y7,    0],
 [ 1,    0, 1, 0, 0, 1,y7,    0, y7,  0,    0,   y7],
 [ 1,    0, 0, 1, 1, 0,y7,    0,  0, y7,   y7,    0],
 [ 1,    0, 0, 1, 0, 1,y7,    0,  0, y7,    0,   y7],
 [ 0,    1, 1, 0,y4, 0, 0,   y7, y7,  0,y7*y4,    0],
 [ 0,    1, 1, 0, 0,y4, 0,   y7, y7,  0,    0,y7*y4],
 [ 0,    1, 0, 1,y4, 0, 0,   y7,  0, y7,y7*y4,    0],
 [ 0,    1, 0, 1, 0,y4, 0,   y7,  0, y7,    0,y7*y4],
 [ 0,    0, 0, 0, 0, 0, 1,y2*y3, y2, y3,   y5,   y6],
 [ 1,y2*y3,y2,y3,y5,y6, 0,    0,  0,  0,    0,    0]])
J=ideal(M.minors(6));
JJ=ideal(y2*y3*y4*y5*y6*y7)
KK=J.saturation(JJ)
KK[0].primary_decomposition();
```

```
[Ideal (y3 - 1, y2*y4 - y5 - y6 + 1) of Multivariate Polynomial Ring in y2, y3, y4, y5,
y6, y7 over Rational Field, Ideal (y2 - 1, y3*y4 - y5 - y6 + 1) of Multivariate Polynomial
Ring in y2, y3, y4, y5, y6, y7 over Rational Field]
```

```
# 1)

# for y3 - 1, y2*y4 - y5 - y6 + 1 => y5*y6 = y2*y4 the slack matrix has \
   the form

R.<y4,y5,y6,y7>=QQ[];

M = matrix(R,[
 [ 1,    0,   y4, 0, 1, 0,y7,    0,y4*y7,  0,   y7,    0],
 [ 1,    0,   y4, 0, 0, 1,y7,    0,y4*y7,  0,    0,   y7],
 [ 1,    0,    0, 1, 1, 0,y7,    0,    0, y7,   y7,    0],
 [ 1,    0,    0, 1, 0, 1,y7,    0,    0, y7,    0,   y7],
 [ 0,   y4,   y4, 0,y4, 0, 0,y4*y7,y4*y7,  0,y7*y4,    0],
 [ 0,   y4,   y4, 0, 0,y4, 0,y4*y7,y4*y7,  0,    0,y7*y4],
 [ 0,   y4,    0, 1,y4, 0, 0,y4*y7,    0, y7,y7*y4,    0],
 [ 0,   y4,    0, 1, 0,y4, 0,y4*y7,    0, y7,    0,y7*y4],
 [ 0,    0,    0, 0, 0, 0, 1,y5*y6,y5*y6,  1,   y5,   y6],
 [ 1,y5*y6,y5*y6, 1,y5,y6, 0,    0,    0,  0,    0,    0]])
J=ideal(M.minors(6));
JJ=ideal(y4*y5*y6*y7)
KK=J.saturation(JJ)
KK
```

```
(Ideal (y5*y6 - y5 - y6 + 1) of Multivariate Polynomial Ring in y4, y5, y6, y7 over
Rational Field, 1)
```

```
# 1)

# results
```

```
# thus for y3 - 1, y2*y4 - y5 - y6 + 1 => y5*y6 = y2*y4 and y6 = 1 the \
   slack matrix has the form

M = matrix(R,[
 [ 1,    0,   y4, 0, 1, 0,y7,    0,y4*y7,  0,   y7,    0],
 [ 1,    0,   y4, 0, 0, 1,y7,    0,y4*y7,  0,    0,   y7],
 [ 1,    0,    0, 1, 1, 0,y7,    0,    0, y7,   y7,    0],
 [ 1,    0,    0, 1, 0, 1,y7,    0,    0, y7,    0,   y7],
 [ 0,   y4,   y4, 0,y4, 0, 0,y4*y7,y4*y7,  0,y7*y4,    0],
 [ 0,   y4,   y4, 0, 0,y4, 0,y4*y7,y4*y7,  0,    0,y7*y4],
 [ 0,   y4,    0, 1,y4, 0, 0,y4*y7,    0, y7,y7*y4,    0],
 [ 0,   y4,    0, 1, 0,y4, 0,y4*y7,    0, y7,    0,y7*y4],
 [ 0,    0,    0, 0, 0, 0, 1,   y5,   y5,  1,   y5,    1],
 [ 1,   y5,   y5, 1,y5, 1, 0,    0,    0,  0,    0,    0]])

(diagonal_matrix([1,1,1,1,1,1,1,1,y7,1])*M*diagonal_matrix([1,1/y4,1/y4\
   ,1,1/y4,1/y4,1/(y7),1/(y7*y4),1/(y7*y4),1/(y7),1/(y7*y4),1/(y7*y4)])).\
   matrix_from_rows_and_columns\
   ([6,7,4,5,2,3,0,1,8,9],[1,0,3,2,4,5,7,6,9,8,10,11])
```

```
[    1     0     1     0     1     0     1     0     1     0     1     0]
[    1     0     1     0     0     1     1     0     1     0     0     1]
[    1     0     0     1     1     0     1     0     0     1     1     0]
[    1     0     0     1     0     1     1     0     0     1     0     1]
[    0     1     1     0  1/y4     0     0     1     1     0  1/y4     0]
[    0     1     1     0     0  1/y4     0     1     1     0     0  1/y4]
[    0     1     0     1  1/y4     0     0     1     0     1  1/y4     0]
[    0     1     0     1     0  1/y4     0     1     0     1     0  1/y4]
[    0     0     0     0     0     0 y5/y4     1     1 y5/y4 y5/y4  1/y4]
[y5/y4     1     1 y5/y4 y5/y4  1/y4     0     0     0     0     0     0]
```

```
# thus for y3 - 1, y2*y4 - y5 - y6 + 1 => y5*y6 = y2*y4 and y5 = 1 the \
   slack matrix has the form

M = matrix(R,[
 [ 1,    0,   y4, 0, 1, 0,y7,    0,y4*y7,  0,   y7,    0],
 [ 1,    0,   y4, 0, 0, 1,y7,    0,y4*y7,  0,    0,   y7],
 [ 1,    0,    0, 1, 1, 0,y7,    0,    0, y7,   y7,    0],
 [ 1,    0,    0, 1, 0, 1,y7,    0,    0, y7,    0,   y7],
 [ 0,   y4,   y4, 0,y4, 0, 0,y4*y7,y4*y7,  0,y7*y4,    0],
 [ 0,   y4,   y4, 0, 0,y4, 0,y4*y7,y4*y7,  0,    0,y7*y4],
 [ 0,   y4,    0, 1,y4, 0, 0,y4*y7,    0, y7,y7*y4,    0],
 [ 0,   y4,    0, 1, 0,y4, 0,y4*y7,    0, y7,    0,y7*y4],
 [ 0,    0,    0, 0, 0, 0, 1,   y6,   y6,  1,    1,   y6],
 [ 1,   y6,   y6, 1, 1,y6, 0,    0,    0,  0,    0,    0]])

(diagonal_matrix([1,1,1,1,1,1,1,1,y7,1])*M*diagonal_matrix([1,1/y4,1/y4\
   ,1,1/y4,1/y4,1/(y7),1/(y7*y4),1/(y7*y4),1/(y7),1/(y7*y4),1/(y7*y4)])).\
   matrix_from_rows_and_columns\
   ([7,6,5,4,3,2,1,0,8,9],[1,0,3,2,5,4,7,6,9,8,11,10])
```

```
[    1     0     1     0     1     0     1     0     1     0     1     0]
```

```
[    1     0     1     0     0     1     1     0     1     0     0     1]
[    1     0     0     1     1     0     1     0     0     1     1     0]
[    1     0     0     1     0     1     1     0     0     1     0     1]
[    0     1     1     0  1/y4     0     0     1     1     0  1/y4     0]
[    0     1     1     0     0  1/y4     0     1     1     0     0  1/y4]
[    0     1     0     1  1/y4     0     0     1     0     1  1/y4     0]
[    0     1     0     1     0  1/y4     0     1     0     1     0  1/y4]
[    0     0     0     0     0     0 y6/y4     1     1 y6/y4 y6/y4  1/y4]
[y6/y4     1     1 y6/y4 y6/y4  1/y4     0     0     0     0     0     0]
```

```
# 1)

# for y2 - 1, y3*y4 - y5 - y6 + 1 => y5*y6 = y3*y4 the slack matrix has \
   the form

R.<y4,y5,y6,y7>=QQ[];

M = matrix(R,[
 [ 1,    0, 1,    0, 1, 0,y7,    0, y7,    0,   y7,    0],
 [ 1,    0, 1,    0, 0, 1,y7,    0, y7,    0,    0,   y7],
 [ 1,    0, 0,   y4, 1, 0,y7,    0,  0,y7*y4,   y7,    0],
 [ 1,    0, 0,   y4, 0, 1,y7,    0,  0,y7*y4,    0,   y7],
 [ 0,   y4, 1,    0,y4, 0, 0,y7*y4, y7,    0,y7*y4,    0],
 [ 0,   y4, 1,    0, 0,y4, 0,y7*y4, y7,    0,    0,y7*y4],
 [ 0,   y4, 0,   y4,y4, 0, 0,y7*y4,  0,y7*y4,y7*y4,    0],
 [ 0,   y4, 0,   y4, 0,y4, 0,y7*y4,  0,y7*y4,    0,y7*y4],
 [ 0,    0, 0,    0, 0, 0, 1,y5*y6,  1,y5*y6,   y5,   y6],
 [ 1,y5*y6, 1,y5*y6,y5,y6, 0,    0,  0,    0,    0,    0]])
J=ideal(M.minors(6));
JJ=ideal(y4*y5*y6*y7)
KK=J.saturation(JJ)
KK
```

```
(Ideal (y5*y6 - y5 - y6 + 1) of Multivariate Polynomial Ring in y4, y5, y6, y7 over
Rational Field, 1)
```

```
# 1)

# results

# thus for y2 - 1, y3*y4 - y5 - y6 + 1 => y5*y6 = y3*y4 and y6 = 1 the \
   slack matrix has the form

M = matrix(R,[
 [ 1,    0, 1,    0, 1, 0,y7,    0, y7,    0,   y7,    0],
 [ 1,    0, 1,    0, 0, 1,y7,    0, y7,    0,    0,   y7],
 [ 1,    0, 0,   y4, 1, 0,y7,    0,  0,y7*y4,   y7,    0],
 [ 1,    0, 0,   y4, 0, 1,y7,    0,  0,y7*y4,    0,   y7],
 [ 0,   y4, 1,    0,y4, 0, 0,y7*y4, y7,    0,y7*y4,    0],
 [ 0,   y4, 1,    0, 0,y4, 0,y7*y4, y7,    0,    0,y7*y4],
 [ 0,   y4, 0,   y4,y4, 0, 0,y7*y4,  0,y7*y4,y7*y4,    0],
 [ 0,   y4, 0,   y4, 0,y4, 0,y7*y4,  0,y7*y4,    0,y7*y4],
```

```
 [ 0,    0, 0,    0, 0, 0, 1,   y5,  1,   y5,   y5,    1],
 [ 1,   y5, 1,   y5,y5, 1, 0,    0,  0,    0,    0,    0]])

(diagonal_matrix([1,1,1,1,1,1,1,1,y7,1])*M*diagonal_matrix([1,1/y4,1,1/y4\
   ,1/y4,1/y4,1/(y7),1/(y7*y4),1/y7,1/(y7*y4),1/(y7*y4),1/(y7*y4)])).\
   matrix_from_rows_and_columns\
   ([4,5,6,7,0,1,2,3,8,9],[1,0,2,3,4,5,7,6,8,9,10,11])
# thus for y2 - 1, y3*y4 - y5 - y6 + 1 => y5*y6 = y3*y4 and y5 = 1 the \
   slack matrix has the form
[   1    0    1    0    1    0    1    0    1    0    1    0]
[   1    0    1    0    0    1    1    0    1    0    0    1]
[   1    0    0    1    1    0    1    0    0    1    1    0]
[   1    0    0    1    0    1    1    0    0    1    0    1]
[   0    1    1    0  1/y4    0    0    1    1    0  1/y4    0]
[   0    1    1    0    0  1/y4    0    1    1    0    0  1/y4]
[   0    1    0    1  1/y4    0    0    1    0    1  1/y4    0]
[   0    1    0    1    0  1/y4    0    1    0    1    0  1/y4]
[   0    0    0    0    0    0 y5/y4    1    1 y5/y4 y5/y4  1/y4]
[y5/y4    1    1 y5/y4 y5/y4  1/y4    0    0    0    0    0    0]
```

```
M = matrix(R,[
 [ 1,    0, 1,    0, 1, 0,y7,    0, y7,    0,   y7,    0],
 [ 1,    0, 1,    0, 0, 1,y7,    0, y7,    0,    0,   y7],
 [ 1,    0, 0,   y4, 1, 0,y7,    0,  0,y7*y4,   y7,    0],
 [ 1,    0, 0,   y4, 0, 1,y7,    0,  0,y7*y4,    0,   y7],
 [ 0,   y4, 1,    0,y4, 0, 0,y7*y4, y7,    0,y7*y4,    0],
 [ 0,   y4, 1,    0, 0,y4, 0,y7*y4, y7,    0,    0,y7*y4],
 [ 0,   y4, 0,   y4,y4, 0, 0,y7*y4,  0,y7*y4,y7*y4,    0],
 [ 0,   y4, 0,   y4, 0,y4, 0,y7*y4,  0,y7*y4,    0,y7*y4],
 [ 0,    0, 0,    0, 0, 0, 1,   y6,  1,   y6,    1,   y6],
 [ 1,   y6, 1,   y6, 1,y6, 0,    0,  0,    0,    0,    0]])

(diagonal_matrix([1,1,1,1,1,1,1,1,y7,1])*M*diagonal_matrix([1,1/y4,1,1/y4\
   ,1/y4,1/y4,1/(y7),1/(y7*y4),1/y7,1/(y7*y4),1/(y7*y4),1/(y7*y4)])).\
   matrix_from_rows_and_columns\
   ([5,4,7,6,1,0,3,2,8,9],[1,0,2,3,5,4,7,6,8,9,11,10])
[   1    0    1    0    1    0    1    0    1    0    1    0]
[   1    0    1    0    0    1    1    0    1    0    0    1]
[   1    0    0    1    1    0    1    0    0    1    1    0]
[   1    0    0    1    0    1    1    0    0    1    0    1]
[   0    1    1    0  1/y4    0    0    1    1    0  1/y4    0]
[   0    1    1    0    0  1/y4    0    1    1    0    0  1/y4]
[   0    1    0    1  1/y4    0    0    1    0    1  1/y4    0]
[   0    1    0    1    0  1/y4    0    1    0    1    0  1/y4]
[   0    0    0    0    0    0 y6/y4    1    1 y6/y4 y6/y4  1/y4]
[y6/y4    1    1 y6/y4 y6/y4  1/y4    0    0    0    0    0    0]
```

```
# 2)

R.<y1,y2,y3,y4,y5,y6,y7,y8,y9,y10,y11,y12,y13,y14,y15,y16,y17,y18,y19,y20\
```

```
   >=QQ[];

M = matrix(R,[
 [ 1, 0, 1, 0, 1, 0, y7,  0, y7,  0,     y7,       0],
 [ 1, 0, 1, 0, 0, 1, y8,  0, y8,  0,      0,      y8],
 [ 1, 0, 0, 1, 1, 0, y9,  0,  0, y9, y9*y12,       0],
 [ 1, 0, 0, 1, 0, 1,y10,  0,  0,y10,      0,y10*y12],
 [ 0, 1, 1, 0,y4, 0,  0,y11,y11,  0,    y11,       0],
 [ 0, 1, 1, 0, 0,y4,  0,y13,y13,  0,      0,     y13],
 [ 0, 1, 0, 1,y4, 0,  0,y14,  0,y14,y14*y12,       0],
 [ 0, 1, 0, 1, 0,y4,  0,y15,  0,y15,      0,y15*y12],
 [ 0, 0, 0, 0, 0, 0,  1,y16,y17,y18,    y19,     y20],
 [ 1,y1,y2,y3,y5,y6,  0,  0,  0,  0,      0,       0]])
K=M.matrix_from_rows([0,1,2,3,4,5,6,7,8])
J=ideal(K.minors(6));
JJ=ideal(y1*y2*y3*y4*y5*y6*y7*y8*y9*y10*y11*y12*y13*y14*y15*y16*y17*y18*\
   y19*y20)
KK=J.saturation(JJ)
KK
```

```
(Ideal (y14 - y15, y13 - y15, y12 - 1, y11 - y15, y10 - y15, y9 - y15, y8 - y15, y7 - y15,
y4 - 1) of Multivariate Polynomial Ring in y1, y2, y3, y4, y5, y6, y7, y8, y9, y10, y11,
y12, y13, y14, y15, y16, y17, y18, y19, y20 over Rational Field, 1)
```

```
# since y4 - 1, y12 - 1 ae in the ideal the results in the case 2) are \
   included in the case 1)
```